

Eigenvalues and structured properties of P-tensors

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Abstract

We define two new constants associated with real eigenvalues of a P-tensor. With the help of these two constants, in the case of P-tensors, we establish upper bounds of two important quantities, whose positivity is a necessary and sufficient condition for a general tensor to be a P-tensor.

Key words: P-tensor, Complementarity problem, Eigenvalues.

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1 Introduction

Let $A = (a_{ij})$ be an $n \times n$ real matrix and $\mathbf{q} \in \mathbb{R}^n$. Then the linear complementarity problem, denoted by LCP (A, \mathbf{q}) , is to find $\mathbf{x} \in \mathbb{R}^n$ such that

$$\text{LCP}(\mathbf{A}, \mathbf{q}) \quad \mathbf{x} \geq \mathbf{0}, \mathbf{q} + A\mathbf{x} \geq \mathbf{0}, \text{ and } \mathbf{x}^\top(\mathbf{q} + A\mathbf{x}) = 0$$

or to show that no such vector exists. It is well-known that the LCP(A, \mathbf{q}) has wide and important applications in engineering and economics (Cottle, Pang and Stone [7] and Han, Xiu and Qi [15]).

In past several decades, there have been a growing literature concerned with the error bounds for LCP (A, \mathbf{q}) . The error bounds for LCP (A, \mathbf{q}) have been given in Chen and

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Xiang [2, 3], Mathias and Pang [22] for P-matrix; Chen, Li, Wu, Vong [4] for MB-matrix; Dai [8] for DB-matrix; Dai, Li, Lu [9, 10] for SB-matrix; García-Esnaola and Peña [12] for B-Matrix; García-Esnaola and Peña [13] for BS-Matrix; García-Esnaola and Peña [14], Li and Zheng [16] for H-Matrix; Luo, Mangasarian, Ren, Solodov [20] for nondegenerate matrix. Recently, Sun and Wang [31] studied the error bounds for generalized linear complementarity problem under some proper assumptions. The componentwise error bounds for LCP (A, \mathbf{q}) was showed by Wang and Yuan [32].

The nonlinear complementarity problem, defined by a nonlinear function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, denoted by NCP(F), is to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{NCP}(F) \quad \mathbf{x} \geq \mathbf{0}, F(\mathbf{x}) \geq \mathbf{0}, \text{ and } \mathbf{x}^\top F(\mathbf{x}) = 0,$$

or to show that no such vector exists. The NCP was introduced by Cottle in his Ph.D. thesis in 1964. the study of NCP(F) have a long history and wide applications in mathematical sciences and applied sciences (Facchinei and Pang [11]). We call the NCP(F) the **tensor complementarity problem**, denoted by TCP $(\mathcal{A}, \mathbf{q})$ iff $F(\mathbf{x}) = \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}$ in the NCP(F), i.e., finding $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{TCP}(\mathcal{A}, \mathbf{q}) \quad \mathbf{x} \geq \mathbf{0}, \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \text{ and } \mathbf{x}^\top (\mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}) = 0$$

or showing that no such vector exists, where $\mathcal{A} = (a_{i_1 \dots i_m})$ is a real m th order n -dimensional tensor (hypermatrix).

The TCP $(\mathcal{A}, \mathbf{q})$ is a natural extension of the LCP (A, \mathbf{q}) . It has some similar properties to the LCP (A, \mathbf{q}) . At the same time, the TCP $(\mathcal{A}, \mathbf{q})$, as a specially structured NCP(F), should have its particular and nice properties other than the general NCP(F). So how to obtain the nice properties and their applications of the TCP $(\mathcal{A}, \mathbf{q})$ will be very interesting by means of the special structure of higher order tensors (hypermatices). Recently, the solution of TCP(\mathcal{A}, \mathbf{q}) and related problems have been well studied. For example, Che, Qi, Wei [5] investigated the existence and uniqueness of solution of TCP $(\mathcal{A}, \mathbf{q})$ for some special tensors. Song and Qi [27, 29] studied the existence of solution of the TCP $(\mathcal{A}, \mathbf{q})$ with the help of the structure of the tensor \mathcal{A} . Song and Yu [30] showed the properties of solution set of the TCP $(\mathcal{A}, \mathbf{q})$. Luo, Qi and Xiu [19] obtained the sparsest solutions to the TCP $(\mathcal{A}, \mathbf{q})$ for Z-tensors. Song and Qi [28], Ling, He, Qi [17, 18], Chen, Yang, Ye [6] studied the the tensor eigenvalue complementarity problem for higher order tensors. See these papers and references therein.

In this paper, we introduce two new constants associated with real eigenvalues of P-tensors. With the help of these two constants, for a P-tensor \mathcal{A} , we establish upper bounds of two quantities $\alpha(F_{\mathcal{A}})$ and $\alpha(T_{\mathcal{A}})$. These two quantities were defined for general tensors and even order tensors respectively by Song and Qi [26]. It was shown there that an m -order

n -dimensional tensor \mathcal{A} is a P-(P_0 -)tensor if and only if $\alpha(T_{\mathcal{A}})$ is positive (nonnegative), and when m is even, \mathcal{A} is a P-(P_0 -)tensor if and only if $\alpha(F_{\mathcal{A}})$ is positive (nonnegative).

The rest of this article is organized as follows. In Section 2, we will give some definitions and basic conclusions, which will be used later on. In Section 3, we will define two new constants associated with real eigenvalues of P-tensors and show their upper and lower bounds. In particular, for a P-tensor \mathcal{A} , we establish upper bounds of $\alpha(F_{\mathcal{A}})$ and $\alpha(T_{\mathcal{A}})$.

We briefly describe our notation. Denote $\mathbb{R}^n := \{(x_1, x_2, \dots, x_n)^T; x_i \in \mathbb{R}, i \in I_n\}$ and $\mathbb{C}^n := \{(x_1, x_2, \dots, x_n)^T; x_i \in \mathbb{C}, i \in I_n\}$, where \mathbb{R} is the set of real numbers. and \mathbb{C} is the set of complex numbers. Denote $I_n := \{1, 2, \dots, n\}$. For any vector $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^{[m-1]}$ is a vector in \mathbb{C}^n with its i th component defined as x_i^{m-1} for $i \in I_n$, and $\mathbf{x} \in \mathbb{R}^n$, \mathbf{x}_+ is a vector in \mathbb{R}^n with $(\mathbf{x}_+)_i = x_i$ if $x_i \geq 0$ and $(\mathbf{x}_+)_i = 0$ if $x_i < 0$ for $i \in I_n$. We assume that $m \geq 2$ and $n \geq 1$. We use small letters x, u, v, α, \dots , for scalars, small bold letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \dots$, for vectors, capital letters A, B, \dots , for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$, for tensors. We denote the zero tensor in $T_{m,n}$ by \mathcal{O} . Denote the set of all real m th order n -dimensional tensors by $T_{m,n}$. We denote by \mathcal{A}_r^J the principal sub-tensor of a tensor $\mathcal{A} \in T_{m,n}$ such that the entries of \mathcal{A}_r^J are indexed by $J \subset I_n$ with $|J| = r$ ($1 \leq r \leq n$), and denote by \mathbf{x}_J the r -dimensional sub-vector of a vector $\mathbf{x} \in \mathbb{R}^n$, with the components of \mathbf{x}_J indexed by J .

2 Preliminaries and basic facts

In this section, we will collect some basic definitions and facts, which will be used later on.

All the tensors discussed in this paper are real. An m -order n -dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \dots i_m})$ is a multi-array of real entries $a_{i_1 \dots i_m}$, where $i_j \in I_n$ for $j \in I_m$. If the entries $a_{i_1 \dots i_m}$ are invariant under any permutation of their indices, then \mathcal{A} is called a **symmetric tensor**.

Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in \mathbb{R}^n with its i th component as

$$(\mathcal{A}\mathbf{x}^{m-1})_i := \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m}$$

for $i \in I_n$. We now give the definitions of P-tensors, which was introduced by Song and Qi [26].

Definition 2.1. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$. We say that A is

- (i) a P_0 tensor iff for any nonzero vector \mathbf{x} in \mathbb{R}^n , there exists $i \in I_n$ such that $x_i \neq 0$ and

$$x_i (\mathcal{A}\mathbf{x}^{m-1})_i \geq 0;$$

(ii) a P tensor iff for any nonzero vector \mathbf{x} in \mathbb{R}^n ,

$$\max_{i \in I_n} x_i (\mathcal{A}\mathbf{x}^{m-1})_i > 0.$$

The concepts of tensor eigenvalues were introduced by Qi [23, 24] to the higher order symmetric tensors, and the existence of the eigenvalues and some applications were studied there. Lim [21] independently introduced real tensor eigenvalues and obtained some existence results using a variational approach.

Definition 2.2. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$. A number $\lambda \in \mathbb{C}$ is called

(i) an **eigenvalue** of \mathcal{A} iff there is a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ such that

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}^{[m-1]}, \quad (2.1)$$

and \mathbf{x} is called an **eigenvector** of \mathcal{A} , associated with λ . An eigenvalue λ corresponding a real eigenvector \mathbf{x} is real and is called an **H-eigenvalue**, and \mathbf{x} is called an **H-eigenvector** of \mathcal{A} , respectively;

(ii) an **E-eigenvalue** of \mathcal{A} iff there is a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ such that

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}, \quad \mathbf{x}^\top \mathbf{x} = 1, \quad (2.2)$$

and \mathbf{x} is called an **E-eigenvector** of \mathcal{A} , associated with λ . An E-eigenvalue λ corresponding a real E-eigenvector \mathbf{x} is real and is called a **Z-eigenvalue**, and \mathbf{x} is called a **Z-eigenvector** of \mathcal{A} , respectively.

The concept of principal sub-tensors was introduced and used in [23] for symmetric tensors.

Definition 2.3. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$. A tensor $\mathcal{C} \in T_{m,r}$ is called a **principal sub-tensor** of a tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$ ($1 \leq r \leq n$) iff there is a set J that composed of r elements in I_n such that

$$\mathcal{C} = (a_{i_1 \dots i_m}), \text{ for all } i_1, i_2, \dots, i_m \in J.$$

Denote such a principal sub-tensor \mathcal{C} by \mathcal{A}_r^J .

The following is a basic conclusion in the study of P-tensors.

Lemma 2.1. (Song and Qi [26, Theorem 3.1 and 4.1]) Let $\mathcal{A} \in T_{m,n}$ be a P-tensor. Then

(i) all principal diagonal entries of \mathcal{A} are positive ($a_{ii \dots i} > 0$ for all $i \in I_n$);

- (ii) each principal sub-tensor of \mathcal{A} is a P-tensor;
- (iii) all H-eigenvalues of each principal sub-tensor of \mathcal{A} are positive when m is even;
- (iv) all Z-eigenvalues of each principal sub-tensor of \mathcal{A} are positive when m is even.

Recall that an operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called **positively homogeneous** iff $T(t\mathbf{x}) = tT(\mathbf{x})$ for each $t > 0$ and all $\mathbf{x} \in \mathbb{R}^n$. For $\mathbf{x} \in \mathbb{R}^n$, it is known well that

$$\|\mathbf{x}\|_\infty := \max\{|x_i|; i \in I_n\} \text{ and } \|\mathbf{x}\|_2 := \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}}$$

are two main norms defined on \mathbb{R}^n . Then for a continuous, positively homogeneous operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, it is obvious that

$$\|T\|_\infty := \max_{\|\mathbf{x}\|_\infty=1} \|T(\mathbf{x})\|_\infty$$

is an operator norm of T and $\|T(\mathbf{x})\|_\infty \leq \|T\|_\infty \|\mathbf{x}\|_\infty$ for any $\mathbf{x} \in \mathbb{R}^n$.

Let $\mathcal{A} \in T_{m,n}$. Define an operator $T_{\mathcal{A}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by for any $\mathbf{x} \in \mathbb{R}^n$,

$$T_{\mathcal{A}}(\mathbf{x}) := \begin{cases} \|\mathbf{x}\|_2^{2-m} \mathcal{A} \mathbf{x}^{m-1}, & \mathbf{x} \neq \mathbf{0} \\ \mathbf{0}, & \mathbf{x} = \mathbf{0}. \end{cases} \quad (2.3)$$

When m is even, define another operator $F_{\mathcal{A}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by for any $\mathbf{x} \in \mathbb{R}^n$,

$$F_{\mathcal{A}}(\mathbf{x}) := (\mathcal{A} \mathbf{x}^{m-1})^{\left[\frac{1}{m-1}\right]}. \quad (2.4)$$

Clearly, both $F_{\mathcal{A}}$ and $T_{\mathcal{A}}$ are continuous and positively homogeneous. The following upper bounds of the operator norm were established by Song and Qi [25].

Lemma 2.2. (Song and Qi [25, Theorem 4.3]) Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$. Then

- (i) $\|T_{\mathcal{A}}\|_\infty \leq \max_{i \in I_n} \left(\sum_{i_2, \dots, i_m=1}^n |a_{ii_2 \dots i_m}| \right);$
- (ii) $\|F_{\mathcal{A}}\|_\infty \leq \max_{i \in I_n} \left(\sum_{i_2, \dots, i_m=1}^n |a_{ii_2 \dots i_m}| \right)^{\frac{1}{m-1}}, \text{ when } m \text{ is even.}$

Recently, Song and Qi [26] defined two quantities for a P_0 -tensor \mathcal{A} with the help of the above two operators.

$$\alpha(T_{\mathcal{A}}) := \min_{\|\mathbf{x}\|_\infty=1} \max_{i \in I_n} x_i(T_{\mathcal{A}}(\mathbf{x}))_i \quad (2.5)$$

for any m , and

$$\alpha(F_{\mathcal{A}}) := \min_{\|\mathbf{x}\|_\infty=1} \max_{i \in I_n} x_i(F_{\mathcal{A}}(\mathbf{x}))_i \quad (2.6)$$

when m is even.

The monotonicity and boundedness of two constants $\alpha(T_{\mathcal{A}})$ and $\alpha(F_{\mathcal{A}})$ for a high order tensor \mathcal{A} are showed by Song and Qi [26].

Lemma 2.3. (Song and Qi [26, Theorem 4.3]) Let $\mathcal{A} = (a_{i_1 \dots i_m})$ be a P₀ tensor in $T_{m,n}$. Then

- (i) $\alpha(T_{\mathcal{A}}) \leq \alpha(T_{\mathcal{A}_r^J})$ for all principal sub-tensors \mathcal{A}_r^J ;
- (ii) $\alpha(F_{\mathcal{A}}) \leq \alpha(F_{\mathcal{A}_r^J})$ for all principal sub-tensors \mathcal{A}_r^J , when m is even;
- (iii) $\alpha(T_{\mathcal{A}}) \leq \max_{i \in I_n} \left(\sum_{i_2, \dots, i_m=1}^n |a_{ii_2 \dots i_m}| \right);$
- (iv) $\alpha(F_{\mathcal{A}}) \leq \max_{i \in I_n} \left(\sum_{i_2, \dots, i_m=1}^n |a_{ii_2 \dots i_m}| \right)^{\frac{1}{m-1}}$, when m is even.

The necessary and sufficient conditions for P-tensor based upon $\alpha(F_{\mathcal{A}})$ and $\alpha(T_{\mathcal{A}})$ are obtained by Song and Qi [26].

Lemma 2.4. (Song and Qi [26, Theorem 4.4]) Let $\mathcal{A} \in T_{m,n}$. Then

- (i) \mathcal{A} is a P-tensor if and only if $\alpha(T_{\mathcal{A}}) > 0$;
- (ii) when m is even, \mathcal{A} is a P-tensor if and only if $\alpha(F_{\mathcal{A}}) > 0$.

The following conclusions about the solution of TCP(\mathcal{A}, \mathbf{q}) with P-tensor \mathcal{A} are obtained by Song and Qi [27, 29]

Lemma 2.5. (Song and Qi [27, Corollary 3.3, Theorem 3.4] and [29, Theorem 3.1]) Let $\mathcal{A} \in T_{m,n}$ be a P-tensor. Then the TCP(\mathcal{A}, \mathbf{q}) has a solution have a solution for all $\mathbf{q} \in \mathbb{R}^n$, and has only zero vector solution for $\mathbf{q} \geq \mathbf{0}$.

3 Upper bounds of $\alpha(F_{\mathcal{A}})$ and $\alpha(T_{\mathcal{A}})$

The quantities $\alpha(T_{\mathcal{A}})$ and $\alpha(F_{\mathcal{A}})$ play a fundamental role in the error bound analysis of TCP(\mathbf{q}, \mathcal{A}). The two quantities are in general not easy to compute. However, it is easy to derive some upper bounds for them when \mathcal{A} is a P-tensor. Recently, Song and Qi [26] obtained the monotonicity and boundedness of two constants $\alpha(T_{\mathcal{A}})$ and $\alpha(F_{\mathcal{A}})$ for a P-tensor \mathcal{A} . In this section, we will establish some smaller upper bounds. For this purpose, we introduce two quantities about a P-tensor \mathcal{A} :

$$\delta_H(\mathcal{A}) := \min\{\lambda_H(\mathcal{A}_r^J); J \subset I_n, r \in I_n\}, \quad (3.1)$$

where $\lambda_H(\mathcal{A})$ denotes the smallest of H-eigenvalues (if any exists) of a P-tensor \mathcal{A} ;

$$\delta_Z(\mathcal{A}) := \min\{\lambda_Z(\mathcal{A}_r^J); J \subset I_n, r \in I_n\}, \quad (3.2)$$

where $\lambda_Z(\mathcal{A})$ denotes the smallest of Z-eigenvalues (if any exists) of a P-tensor \mathcal{A} . The above two minimum ranges over those principal sub-tensors of \mathcal{A} which indeed have H-eigenvalues (Z-eigenvalues).

It follows from Lemma 2.1 that all principal diagonal entries of \mathcal{A} are positive and all H-(Z-)eigenvalues of each principal sub-tensor of \mathcal{A} are positive when m is even. So $\delta_H(\mathcal{A})$ and $\delta_Z(\mathcal{A})$ are well defined, finite and positive when m is even. Now we give some upper bounds of $\alpha(F_{\mathcal{A}})$ and $\alpha(T_{\mathcal{A}})$ using the quantities $\delta_H(\mathcal{A})$ and $\delta_Z(\mathcal{A})$.

Theorem 3.1. Let $\mathcal{A} \in T_{m,n}$ ($m \geq 2$) be a P-tensor, and let m be an even number. Then

$$\alpha(F_{\mathcal{A}}) \leq (\delta_H(\mathcal{A}))^{\frac{1}{m-1}} \leq (\min_{i \in I_n} a_{ii\dots i})^{\frac{1}{m-1}}. \quad (3.3)$$

Proof. It follows from Lemma 2.1 (i) that

$$a_{ii\dots i} > 0 \text{ for all } i \in I_n.$$

Since $\mathcal{A}_1^J = (a_{ii\dots i})$ ($J = \{i\}$) is m-order 1-dimensional principal sub-tensor of \mathcal{A} , then $a_{ii\dots i}$ is a H-eigenvalue of \mathcal{A}_1^J for all $i \in I_n$, and hence

$$\delta_H(\mathcal{A}) \leq \min_{i \in I_n} a_{ii\dots i}.$$

Next we show the left-hand inequality. Let $\delta = \delta_H(\mathcal{A})$ and $\mathcal{B} = \mathcal{A} - \delta \mathcal{I}$, where \mathcal{I} is unit tensor. Then it follows from the definition of $\delta_H(\mathcal{A})$ that δ is a H-eigenvalue of a principal sub-tensor \mathcal{A}_r^J of \mathcal{A} . Then there exists $\mathbf{x}^* \in \mathbb{R}^r \setminus \{\mathbf{0}\}$ such that

$$(\mathcal{A}_r^J - \delta \mathcal{I}_r^J)(\mathbf{x}^*)^{m-1} = \mathcal{A}_r^J(\mathbf{x}^*)^{m-1} - \delta(\mathbf{x}^*)^{[m-1]} = \mathbf{0}.$$

So the principal sub-tensor $\mathcal{B}_r^J = \mathcal{A}_r^J - \delta \mathcal{I}_r^J$ of \mathcal{B} is not a P-tensor. Thus it follows from Lemma 2.1 (ii) that $\mathcal{B} = \mathcal{A} - \delta \mathcal{I}$ is not a P-tensor. Consequently, there exists a vector \mathbf{y} with $\|\mathbf{y}\|_{\infty} = 1$ such that

$$\max_{i \in I_n} y_i(\mathcal{B}\mathbf{y}^{m-1})_i = \max_{i \in I_n} (y_i(\mathcal{A}\mathbf{y}^{m-1})_i - \delta y_i^m) \leq 0.$$

So, we have

$$y_l(\mathcal{A}\mathbf{y}^{m-1})_l - \delta y_l^m \leq \max_{i \in I_n} (y_i(\mathcal{A}\mathbf{y}^{m-1})_i - \delta y_i^m) \leq 0 \text{ for all } l \in I_n,$$

which implies that for some $j \in I_n$,

$$\max_{i \in I_n} y_i(\mathcal{A}\mathbf{y}^{m-1})_i = y_j(\mathcal{A}\mathbf{y}^{m-1})_j \leq \delta y_j^m \leq \delta \|\mathbf{y}\|_{\infty}^m = \delta.$$

It follows from the definition (Equation (2.6)) of $\alpha(F_{\mathcal{A}})$ that

$$\begin{aligned}\alpha(F_{\mathcal{A}}) &\leq \max_{i \in I_n} y_i(F_{\mathcal{A}}(\mathbf{y}))_i = \max_{i \in I_n} y_i(\mathcal{A}\mathbf{y}^{m-1})_i^{\frac{1}{m-1}} \\ &= \max_{i \in I_n} y_i^{\frac{m-2}{m-1}} (y_i(\mathcal{A}\mathbf{y}^{m-1})_i)^{\frac{1}{m-1}} \\ &\leq \|\mathbf{y}\|_{\infty}^{\frac{m-2}{m-1}} \max_{i \in I_n} (y_i(\mathcal{A}\mathbf{y}^{m-1})_i)^{\frac{1}{m-1}} \\ &\leq \delta^{\frac{1}{m-1}}.\end{aligned}$$

The desired inequality follows. \square

Theorem 3.2. Let $\mathcal{A} \in T_{m,n}$ ($m \geq 2$) be a P-tensor, and let m be an even number. Then

$$\alpha(T_{\mathcal{A}}) \leq \delta_Z(\mathcal{A}) \leq \min_{i \in I_n} a_{ii\dots i}. \quad (3.4)$$

Proof. Using similar proof technique of Theorem 3.1, we have the right-hand inequality holds. Next we show the left-hand inequality. Let $\delta = \delta_Z(\mathcal{A})$ and $\mathcal{B} = \mathcal{A} - \delta\mathcal{E}$, where $\mathcal{E} = I_2^{\frac{m}{2}}$ and I_2 is $n \times n$ unit matrix ($\mathcal{E}\mathbf{x}^{m-1} = \|\mathbf{x}\|_2^{m-2}\mathbf{x}$, see Chang, Pearson, Zhang [1]). Then it follows from the definition of $\delta_Z(\mathcal{A})$ that δ is a Z-eigenvalue of a principal sub-tensor \mathcal{A}_r^J of \mathcal{A} . Then there exists $\mathbf{x}^* \in \mathbb{R}^r \setminus \{\mathbf{0}\}$ such that $(\mathbf{x}^*)^\top \mathbf{x}^* = 1$ and

$$(\mathcal{A}_r^J - \delta\mathcal{E}_r^J)(\mathbf{x}^*)^{m-1} = \mathcal{A}_r^J(\mathbf{x}^*)^{m-1} - \delta\mathbf{x}^* = \mathbf{0}.$$

So the principal sub-tensor $\mathcal{B}_r^J = \mathcal{A}_r^J - \delta\mathcal{E}_r^J$ of \mathcal{B} is not a P-tensor. Thus it follows from Lemma 2.1 (ii) that $\mathcal{B} = \mathcal{A} - \delta\mathcal{E}$ is not a P-tensor. Consequently, there exists a vector \mathbf{y} with $\|\mathbf{y}\|_{\infty} = 1$ such that

$$\max_{i \in I_n} y_i(\mathcal{B}\mathbf{y}^{m-1})_i = \max_{i \in I_n} (y_i(\mathcal{A}\mathbf{y}^{m-1})_i - \delta\|\mathbf{y}\|_2^{m-2}y_i^2) \leq 0.$$

So, we have

$$y_l(\mathcal{A}\mathbf{y}^{m-1})_l - \delta\|\mathbf{y}\|_2^{m-2}y_l^2 \leq \max_{i \in I_n} (y_i(\mathcal{A}\mathbf{y}^{m-1})_i - \delta\|\mathbf{y}\|_2^{m-2}y_i^2) \leq 0 \text{ for all } l \in I_n,$$

which implies that for some $j \in I_n$,

$$\max_{i \in I_n} y_i(\mathcal{A}\mathbf{y}^{m-1})_i = y_j(\mathcal{A}\mathbf{y}^{m-1})_j \leq \delta\|\mathbf{y}\|_2^{m-2}y_j^2 \leq \delta\|\mathbf{y}\|_2^{m-2}\|\mathbf{y}\|_{\infty}^2 = \|\mathbf{y}\|_2^{m-2}\delta.$$

It follows from the definition (Equation (2.5)) of $\alpha(T_{\mathcal{A}})$ that

$$\begin{aligned}\alpha(T_{\mathcal{A}}) &\leq \max_{i \in I_n} y_i(T_{\mathcal{A}}(\mathbf{y}))_i = \max_{i \in I_n} y_i(\|\mathbf{y}\|_2^{2-m}\mathcal{A}\mathbf{y}^{m-1})_i \\ &= \|\mathbf{y}\|_2^{2-m} \max_{i \in I_n} y_i(\mathcal{A}\mathbf{y}^{m-1})_i \\ &\leq \|\mathbf{y}\|_2^{2-m}\|\mathbf{y}\|_2^{m-2}\delta \\ &= \delta.\end{aligned}$$

The desired inequality follows. \square

Question 1. It is known from Lemma 2.4 and Theorem 3.1,3.2 that for a P-tensor \mathcal{A} ,

$$(\min_{i \in I_n} a_{ii\cdots i})^{\frac{1}{m-1}} \geq \alpha(F_{\mathcal{A}}) > 0 \text{ and } \min_{i \in I_n} a_{ii\cdots i} \geq \alpha(T_{\mathcal{A}}) > 0.$$

Then we have the following questions for further research.

- (i) Do two constants $\alpha(F_{\mathcal{A}})$ and $\alpha(T_{\mathcal{A}})$ have a positive lower bound?
- (ii) Are the above upper bounds the smallest?

4 Conclusions

In this paper, We introduce two quantities about a P-tensor \mathcal{A} by means of H- and Z-eigenvalues of real tensors:

$$\begin{aligned}\delta_H(\mathcal{A}) &:= \min\{\lambda_H(\mathcal{A}_r^J); J \subset I_n, r \in I_n\}, \\ \delta_Z(\mathcal{A}) &:= \min\{\lambda_Z(\mathcal{A}_r^J); J \subset I_n, r \in I_n\}.\end{aligned}$$

The upper bounds are obtained, which only depend on the diagonal entries of tensor.

- (iii) $\alpha(F_{\mathcal{A}}) \leq (\delta_H(\mathcal{A}))^{\frac{1}{m-1}} \leq (\min_{i \in I_n} a_{ii\cdots i})^{\frac{1}{m-1}}$ when m is even.
- (iv) $\alpha(T_{\mathcal{A}}) \leq \delta_Z(\mathcal{A}) \leq \min_{i \in I_n} a_{ii\cdots i}$ when m is even.

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